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# A note on the well-posedness of the nonlocal boundary value problem for elliptic difference equations

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**Abstract**

The nonlocal boundary value problem for elliptic difference equations in an arbitrary Banach space is considered. The well-posedness of this problem is investigated. The stability, almost coercive stability and coercive stability estimates for the solutions of difference schemes of the second order of accuracy for the approximate solutions of the nonlocal boundary value problem for elliptic equation are obtained. The theoretical statements for the solution of these difference schemes are supported by the results of numerical experiments.


*Keywords:* Elliptic difference equation; Difference schemes; Well-posedness

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## 1. Introduction. The nonlocal difference problem

Coercivity inequalities in Hölder norms with a weight for the solutions of an abstract differential equation of elliptic type were established for the first time in [1]. Further in [2–11,14–19] the coercive inequalities in Hölder norms with a weight and without a weight were obtained for the solutions of various local and nonlocal boundary value problems for differential and difference equations of elliptic type. In the present paper we consider the nonlocal boundary value problem

$$\begin{cases} -\frac{1}{\tau^2}(u_{k+1}-2u_k+u_{k-1})+Au_k=\varphi_k, \quad 1\leqslant k\leqslant N-1, \\ u_0=u_N, \quad -u_2+4u_1-3u_0=u_{N-2}-4u_{N-1}+3u_N, \quad N\tau=1 \end{cases} \tag{1}$$

for elliptic difference equation in an arbitrary Banach space $E$ with a positive operator $A$.

It is known (see [3]) that for a positive operator $A$ it follows that $B=\frac{1}{2}(\tau A+\sqrt{A(4+\tau^2A)})$ is strongly positive and $R=(I+\tau B)^{-1}$ which is defined on the whole space $E$ is a bounded operator. Furthermore, we have that

$$\|R^k\|_{E\to E}\leqslant M(1+\delta\tau)^{-k}, \quad k\tau\|BR^k\|_{E\to E}\leqslant M, \quad k\geqslant 1, \ \delta>0, \tag{2}$$

$$\|B^\beta(R^{k+r}-R^k)\|_{E\to E}\leqslant M(r\tau)^\alpha(k\tau)^{-\alpha-\beta}, \tag{3}$$

$$1\leqslant k<k+r\leqslant N, \quad 0\leqslant\alpha, \ \beta\leqslant 1,$$

where $M$ does not depend on $\tau$.

First of all let us give some lemmas that will be needed below.

**Lemma 1.** *The estimates hold*:

$$\begin{cases} \|(I-R^N)^{-1}\|_{E\to E}\leqslant M, \\ \|(I-(2I-\tau B)(2I+3\tau B)^{-1}R^{N-2})^{-1}\|_{E\to E}\leqslant M, \end{cases} \tag{4}$$

*where $M$ does not depend on $\tau$.*

The proof of this lemma is based on the estimates (2) and (3).

**Lemma 2.** *For any $\varphi_k$, $1\leqslant k\leqslant N-1$ the solution of the problem* (1) *exists and the following formula holds*:

$$u_k=\sum_{j=1}^{N-1}G(k,j)\varphi_j\tau, \quad 0\leqslant k\leqslant N, \tag{5}$$

*where*

$$G(k,1)=G(k,N-1)=\frac{C}{2}[(R^{N-3}(4R-I)+R-4I]B^{-1}(I-DR^{N-2})^{-1}$$

*for $k = 0$ and $k = N$;*

$$G(k,j) = -C(R^2 - 4R + I)(R^{j-2} + R^{N-j-2})(2B)^{-1}(I - DR^{N-2})^{-1}$$

*for $2 \leqslant j \leqslant N-2$ and $k = 0$, $k = N$;*

$$\begin{aligned} G(k,1) = CC_1(2B)^{-1}\{&R^{k-1}(2(R+3I) + R^2(R-3I)) \\ &+ R^{N-k}(4I - R)(I+R) + R^{N+k-3}(I - 4R)(I+R) \\ &+ R^{2N-k-3}(3R - I - 2R^2(3R+I))\}(I - R^N)^{-1}(I - DR^{N-2})^{-1}, \end{aligned}$$

$$\begin{aligned} G(k,N-1) = -CC_1(2B)^{-1}\{&R^k(R-4I)(R+I) + R^{N-k-1}(-2(R+3I) \\ &+ R^2(3I - R)) + R^{N+k-3}(I - 3R + 2R^2(3R+I)) \\ &+ R^{2N-k-3}(4R - I)(R+I)\}(I - R^N)^{-1}(I - DR^{N-2})^{-1}, \end{aligned}$$

$$\begin{aligned} G(k,j) = CC_1(2B)^{-1}\{&(R-I)^3(R^{j+k-2} + R^{2N-2-j-k}) + (-I + 3R \\ &+ R^2(3I - R))(R^{N-k+j-2} + R^{N+k-j-2}) + 2(I - 3R)(R^{2N-2+j-k} \\ &+ R^{2N-2-j+k}) + 2R^{|j-k|}(R^N - I)(R - 3I + R^{N-2}(-I + 3R))\} \\ &\times (I - R^N)^{-1}(I - DR^{N-2})^{-1} \end{aligned}$$

*for $2 \leqslant j \leqslant N-2$ and $1 \leqslant k \leqslant N-1$. Here*

$$C = (I + \tau B)(2I + 3\tau B)^{-1}, \quad C_1 = (I + \tau B)(2I + \tau B)^{-1},$$
$$D = (2I - \tau B)(2I + 3\tau B)^{-1},$$

*where $I$ is the unit operator.*

**Proof.** We see that the problem (1) can be obviously rewritten as the equivalent nonlocal boundary value problem for the first order linear difference equations

$$\begin{cases} \dfrac{u_k - u_{k-1}}{\tau} + Bu_k = z_k, \quad 1 \leqslant k \leqslant N, \\ u_N = u_0, \quad -u_2 + 4u_1 - 3u_0 = u_{N-2} - 4u_{N-1} + 3u_N, \\ -\dfrac{z_{k+1} - z_k}{\tau} + Bz_k = (1 + \tau B)\varphi_k, \quad 1 \leqslant k \leqslant N-1. \end{cases}$$

From that there follows the system of recursion formulas

$$\begin{cases} u_k = Ru_{k-1} + \tau R z_k, \quad 1 \leqslant k \leqslant N, \\ z_k = Rz_{k+1} + \tau\varphi_k, \quad 1 \leqslant k \leqslant N-1. \end{cases}$$

Hence

$$\begin{cases} u_k = R^k u_0 + \displaystyle\sum_{i=1}^{k} R^{k-i+1}\tau z_i, \quad 1 \leqslant k \leqslant N, \\ z_k = R^{N-k} z_N + \displaystyle\sum_{j=k}^{N-1} R^{j-k}\tau\varphi_j, \quad 1 \leqslant k \leqslant N-1. \end{cases}$$

From the first formula and the condition $u_N = u_0$ it follows that

$$u_N = R^N u_0 + \sum_{i=1}^{N} R^{N-i+1}\tau z_i$$

and

$$\begin{aligned} u_N = u_0 &= (I-R^N)^{-1}\sum_{i=1}^{N} R^{N-i+1}\tau z_i = \frac{1}{1-R^N}\left\{\tau R z_N + \sum_{i=1}^{N-1} R^{N-i+1}\tau z_i\right\} \\ &= (I-R^N)^{-1}\left\{\left(\tau R + \sum_{i=1}^{N-1} R^{2N-2i+1}\tau\right) z_N + \sum_{i=1}^{N-1}\tau R^{N-i+1}\sum_{j=i}^{N-1} R^{j-i}\tau\varphi_j\right\} \\ &= (I-R^N)^{-1}\left\{(R-R^{2N+1})(I-R^2)^{-1}\tau z_N + \sum_{j=1}^{N-1}\tau^2\sum_{i=1}^{j} R^{N+j-2i+1}\varphi_j\right\} \\ &= (I-R^N)^{-1}(I-R^2)^{-1}\left[R(1-R^{2N})\tau z_N + \sum_{j=1}^{N-1}\tau^2[R^{N-j+1}-R^{N+j+1}]\varphi_j\right], \end{aligned} \tag{6}$$

and for $k$, $1 \leqslant k \leqslant N-1$:

$$\begin{aligned} u_k &= (I-R^N)^{-1}\left\{hR^{k+1}z_N + \sum_{i=1}^{N-1} R^{k+N-i+1}hz_i\right\} + \sum_{i=1}^{k} R^{k-i+1}\tau z_i \\ &= (I-R^N)^{-1}(I-R^2)^{-1}R^k\left\{(R-R^{2N+1})(I-R^2)^{-1}\tau z_N + \sum_{j=1}^{N-1}\tau^2\left[R^{N-j+1}-R^{N+j+1}\right]\varphi_j\right\} \\ &\quad + \sum_{i=1}^{k} R^{N+k-2i+1}\tau z_N + \sum_{i=1}^{k}\sum_{j=i}^{N-1}\tau^2 R^{k+j-2i+1}\varphi_j \\ &= (I-R^2)^{-1}\left[R^{k+1}+R^{N-k+1}\right]\tau z_N + (I-R^N)^{-1}(I-R^{N-1})^{-1}\sum_{j=1}^{N-1}\tau^2\left[R^{N-j+1}-R^{N+j+1}\right]\varphi_j \\ &\quad + \sum_{j=1}^{k}\tau^2\sum_{i=1}^{j} R^{k+j-2i+1}f_j + \sum_{j=k+1}^{N-1}\tau^2\sum_{i=1}^{k} R^{k+j-2i+1}\varphi_j \\ &= (I-R^2)^{-1}\left[R^{k+1}+R^{N-k+1}\right]\tau z_N + (I-R^N)^{-1}(I-R^2)^{-1}R^k\sum_{j=1}^{N-1}\tau^2\left[R^{N-j+1}-R^{N+j+1}\right]\varphi_j \\ &\quad + (I-R^2)^{-1}\sum_{j=1}^{N-1}\tau^2\left(R^{|k-j|+1}-R^{k+j+1}\right)\varphi_j. \end{aligned} \tag{7}$$

By using the formulas (6), (7), and the condition $-u_2 + 4u_1 - 3u_0 = u_{N-2} - 4u_{N-1} + 3u_N$, we obtain (5). Lemma 2 is proved. □

## 2. Well-posedness of the nonlocal difference problem

Let $F_\tau(E)$ be the linear space of mesh functions $\varphi^\tau = \{\varphi_k\}_1^{N-1}$ with values in the Banach space $E$. Next on $F_\tau(E)$ we denote by $C_\tau(E)$ and $C_\tau^\alpha(E)$ Banach spaces with the norms

$$\|\varphi^\tau\|_{C_\tau(E)} = \max_{1\leqslant k\leqslant N-1} \|\varphi_k\|_E,$$

$$\|\varphi^\tau\|_{C_\tau^\alpha(E)} = \|\varphi^\tau\|_{C_\tau(E)} + \max_{1\leqslant k<k+r\leqslant N-1} \|\varphi_{k+r} - \varphi_k\|_E \frac{1}{(r\tau)^\alpha}.$$

The nonlocal boundary value problem (1) is said to be stable in $F_\tau(E)$ if we have the inequality

$$\|u^\tau\|_{F_\tau(E)} \leqslant M\|\varphi^\tau\|_{F_\tau(E)},$$

where $M$ is independent not only of $\varphi^\tau$ but also of $\tau$.

We denote $E_\alpha = E_\alpha(B, E)$ as the fractional spaces consisting of all $v \in E$ for which the following norm is finite:

$$\|v\|_{E_\alpha} = \sup_{\lambda>0} \lambda^\alpha \|B(\lambda + B)^{-1} v\|_E.$$

**Theorem 1.** *The nonlocal boundary value problem* (1) *is stable in* $C_\tau(E)$ *norm.*

**Proof.** By [2],

$$\|u^\tau\|_{C_\tau(E)} \leqslant M[\|\varphi\|_E\| + \|\psi\|_E + \|\varphi^\tau\|_{C_\tau(E)}],$$

for the solutions of the boundary value problem

$$\begin{cases} -\frac{1}{\tau^2}[u_{k+1} - 2u_k + u_{k-1}] + Au_k = \varphi_k, \\ 1 \leqslant k \leqslant N-1, \quad u_0 = \varphi, \quad u_N = \psi \end{cases} \tag{8}$$

of the elliptic difference equations in an arbitrary Banach space $E$ with a positive operator $A$. Using the estimates (2)–(4) and the formula (5), we obtain

$$\|u_0\|_E \leqslant M_1\|\varphi^\tau\|_{C_\tau(E)}.$$

Hence, we obtain an estimate of the form

$$\|u^\tau\|_{C_\tau(E)} \leqslant M_2\|\varphi^\tau\|_{C_\tau(E)}.$$

Theorem 1 is proved. □

The nonlocal boundary value problem (1) is said to be coercively stable (well posed) in $F_\tau(E)$ if we have the coercive inequality

$$\|\{\tau^{-2}(u_{k+1} - 2u_k + u_{k-1})\}_1^{N-1}\|_{F_\tau(E)} \leqslant M\|\varphi^\tau\|_{F_\tau(E)},$$

where $M$ is independent not only of $\varphi^\tau$ but also of $\tau$.

Since the nonlocal boundary value problem

$$-u''(t) + Au(t) = f(t) \quad (0 \leqslant t \leqslant 1), \;\; u(0) = u(1), \;\; u'(0) = u'(1) \tag{9}$$

in the space $C(E)$ of continuous functions defined on $[0, 1]$ and with values in $E$ is not well posed for the general positive operator $A$ and space $E$, then the well-posedness of the difference nonlocal boundary value in $C_\tau(E)$ norm does not take place uniformly with respect to $\tau > 0$. This means that the coercive norm

$$\|u^\tau\|_{K_\tau(E)} = \|\{\tau^{-2}(u_{k+1} - 2u_k + u_{k-1})\}_1^{N-1}\|_{C(\tau,E)} + \|Au^\tau\|_{C_\tau(E)}$$

tends to $\infty$ as $\tau \to +0$ . The investigation of the difference problem (1) permits us to establish the order of growth of this norm to $\infty$.

**Theorem 2.** *For the solution of the difference problem* (1) *we have almost coercive inequality*

$$\|u^\tau\|_{K_\tau(E)} \leqslant M \min\left\{\ln\frac{1}{\tau}, 1 + |\ln \|B\|_{E\to E}|\right\}\|\varphi^\tau\|_{C_\tau(E)}, \tag{10}$$

*where $M$ does not depend on $\varphi_k$, $1 \leqslant k \leqslant N-1$ and $\tau$.*

**Proof.** By [2],

$$\|u^\tau\|_{K_\tau(E)} \leqslant M\left[\|A\varphi\|_E\| + \|A\psi\|_E + \min\left\{\ln\frac{1}{\tau}, 1 + |\ln\|B\|_{E\to E}|\right\}\|\varphi^\tau\|_{C_\tau(E)}\right],$$

for the solutions of the boundary value problem (8). Using the estimates (2)–(4) and the formula (5), we obtain

$$\|Au_0\|_E \leqslant M_1 \min\left\{\ln\frac{1}{\tau}, 1 + \left|\ln\|B\|_{E\to E}\right|\right\}\|\varphi^\tau\|_{C_\tau(E)}.$$

Hence, from last two estimates it follows (10). Theorem 2 is proved. □

**Theorem 3.** *The nonlocal boundary value problem* (1) *is well posed in $C_\tau(E_\alpha)$.*

**Proof.** By [8],

$$\begin{aligned}&\|\{\tau^{-2}(u_{k+1} - 2u_k + u_{k-1})\}_1^{N-1}\|_{C_\tau(E_\alpha)}\\ &\quad\leqslant M\left[\|A\varphi\|_{E_\alpha}\| + \|A\psi\|_{E_\alpha} + \frac{1}{\alpha(1-\alpha)}\|\varphi^\tau\|_{C_\tau(E_\alpha)}\right]\end{aligned}$$

for the solutions of the boundary value problem (8). Using the estimates (2)–(4) and the formula (5), we obtain

$$\|Au_0\|_{E_\alpha} \leqslant \frac{M_1}{\alpha(1-\alpha)}\|\varphi^\tau\|_{C_\tau(E_\alpha)}.$$

Hence, we obtain an estimate of the form

$$\|\{\tau^{-2}(u_{k+1}-2u_k+u_{k-1})\}_1^{N-1}\|_{C_\tau(E_\alpha)} \leqslant \frac{M_2}{\alpha(1-\alpha)}\|\varphi^\tau\|_{C_\tau(E_\alpha)}.$$

Theorem 3 is proved. □

Note that the coercivity inequality

$$\left\|\left\{\frac{u_{k+1}-2u_k+u_{k-1}}{\tau^2}\right\}_1^{N-1}\right\|_{C_\tau^\alpha} \leqslant \frac{M}{\alpha(1-\alpha)}\|\varphi^\tau\|_{C_\tau^\alpha}$$

fails for the general positive operator $A$ and space $E$. Nevertheless, we have the following result.

**Theorem 4.** *Let $\varphi_{N-1}-\varphi_1 \in E_\alpha$. Then the coercivity inequality holds*:

$$\left\|\left\{\frac{u_{k+1}-2u_k+u_{k-1}}{\tau^2}\right\}_1^{N-1}\right\|_{C_\tau^\alpha} \leqslant M\left[\frac{M}{\alpha^2(1-\alpha)}\|\varphi^\tau\|_{C_\tau^\alpha}+\frac{1}{\alpha}\|\varphi_1-\varphi_{N-1}\|_{E_\alpha}\right], \tag{11}$$

*where $M$ does not depend on $\varphi_k$, $1 \leqslant k \leqslant N-1$, $\alpha$ and $\tau$.*

**Proof.** By [8],

$$\left\|\left\{\frac{u_{k+1}-2u_k+u_{k-1}}{\tau^2}\right\}_1^{N-1}\right\|_{C_\tau^\alpha} \leqslant M\left[\frac{1}{\alpha}(\|A\varphi-\varphi_1\|_{E_\alpha}\|+\|A\psi-\varphi_{N-1}\|_{E_\alpha})+\frac{1}{\alpha(1-\alpha)}\|\varphi^\tau\|_{C_\tau^\alpha}\right]$$

for the solutions of the boundary value problem (8). Using the estimates (2)–(4) and the formula (5), we obtain

$$\|Au_0-\varphi_1\|_{E_\alpha} \leqslant \frac{M_1}{\alpha(1-\alpha)}\|\varphi^\tau\|_{C_\tau^\alpha}.$$

From last two estimates it follows (11). Theorem 4 is proved. □

Note that by passing to the limit for $\tau \to 0$ one can recover Theorems of the paper [11] on the well-posedness of the nonlocal-boundary value problem (9) in the spaces of smooth functions.

Now we consider the applications of Theorems 1–4. We consider the boundary-value problem on the range $\{0 \leqslant y \leqslant 1, x \in \mathscr{R}^n\}$ for elliptic equation

$$\begin{cases} -\frac{\partial^2 u}{\partial y^2} + \sum_{|r|=2m} a_r(x) \frac{\partial^{|\tau|} u}{\partial x_1^{r_1} \cdots \partial x_n^{r_n}} + \delta u(y,x) = f(y,x), \\ 0 < y < 1, \quad x, r \in \mathscr{R}^n, \quad |r| = r_1 + \cdots + r_n, \\ u(0,x) = u(1,x), \quad u_y(0,x) = u_y(1,x), \quad x \in \mathscr{R}^n, \end{cases} \tag{12}$$

where $a_r(x)$ and $f(y, x)$ are given sufficiently smooth functions and $\delta > 0$ is the sufficiently large number.

We will assume that the symbol

$$B^x(\xi) = \sum_{|r|=2m} a_r(x)(\mathrm{i}\xi_1)^{r_1} \cdots (\mathrm{i}\xi_n)^{r_n}, \quad \xi = (\xi_1, \ldots, \xi_n) \in R^n$$

of the differential operator of the form

$$B^x = \sum_{|r|=2m} a_r(x) \frac{\partial^{|r|}}{\partial x_1^{r_1} \cdots \partial x_n^{r_n}} \tag{13}$$

acting on functions defined on the space $\mathscr{R}^n$, satisfies the inequalities

$$0 < M_1 |\xi|^{2m} \leqslant (-1)^m B^x(\xi) \leqslant M_2 |\xi|^{2m} < \infty$$

for $\xi \neq 0$.

The discretization of problem (12) is carried out in two steps. In the first step let us give the difference operator $A_h^x$ by the formula

$$A_h^x u_x^h = \sum_{2m \leqslant |r| \leqslant S} b_r^x D_h^r u_x^h + \delta u_x^h. \tag{14}$$

The coefficients are chosen in such a way that the operator $A_h^x$ approximates in a specified way the operator

$$\sum_{|r|=2m} a_r(x) \frac{\partial^{|r|}}{\partial x_1^{r_1} \cdots \partial x_n^{r_n}} + \delta.$$

We shall assume that for $|\xi_k h| \leqslant \pi$ the symbol $A(\xi h, h)$ of the operator $A_h^x - \delta$ satisfies the inequalities

$$(-1)^m A^x(\xi h, h) \geqslant M_1 |\xi|^{2m}, \quad |\arg A^x(\xi h, h)| \leqslant \phi < \phi_0 < \frac{\pi}{2}. \tag{15}$$

With the help of $A_h^x$ we arrive at the boundary value problem

$$-\frac{\mathrm{d}^2 v^h(y,x)}{\mathrm{d}y^2} + A_h^x v^h(y,x) = \varphi^h(y,x), \quad 0 < y < 1, \tag{16}$$

$$v^h(0,x) = v^h(1,x), \quad v_y^h(0,x) = v_y^h(1,x), \quad x \in \mathscr{R}_h^n,$$

for an infinite system of ordinary differential equations.

In the second step we replace problem (16) by the difference scheme

$$\begin{cases} -\frac{1}{\tau^2}[u_{k+1}^h - 2u_k^h + u_{k-1}^h] + A_h^x u_k^h = \varphi_k^h, \quad 1 \leqslant k \leqslant N-1, \\ u_0^h = u_N^h, \quad -u_2^h + 4u_1^h - 3u_0^h = u_{N-2}^h - 4u_{N-1}^h + 3u_N^h, \quad N\tau = 1. \end{cases} \tag{17}$$

Let us give a number of corollaries of the abstract theorems given in the above.

**Theorem 5.** *Let $\tau$ and $h$ be a sufficiently small numbers. Then the solutions of the difference schemes (17) satisfy the following stability estimates*:

$$\|u^{\tau,h}\|_{C_\tau(C_h^\beta)} \leqslant M\|\varphi^{\tau,h}\|_{C_\tau(C_h^\beta)}, \quad 0 \leqslant \beta < 1,$$

*where $M$ does not depend on $\varphi^{\tau,h}$, $\beta$, $h$ and $\tau$.*

The proof of Theorem 5 is based on the abstract Theorem 1, the positivity of the operator $A_h^x$ in $C_h^\beta$ .

Now, we consider the coercive stability of (17).

**Theorem 6.** *Let $\tau$ and $h$ be a sufficiently small numbers. Then the solutions of the difference schemes (17) satisfy the following almost coercive stability estimates*:

$$\|\{\tau^{-2}(u_{k+1}^h - 2u_k^h + u_{k-1}^h)\}_1^{N-1}\|_{C_\tau(C_h)} \leqslant M \ln\frac{1}{\tau + h}\|\varphi^{\tau,h}\|_{C_\tau(C_h)},$$

*where $M$ does not depend on $\varphi^{\tau,h}$, $h$ and $\tau$.*

The proof of Theorem 6 is based on the abstract Theorem 2, the positivity of the operator $A_h^x$ in $C_h$ and on the estimate

$$\min\left\{\ln\frac{1}{\tau}, 1 + |\ln\|B_h^x\|_{C_h\to C_h}|\right\} \leqslant M\ln\frac{1}{\tau+h}.$$

**Theorem 7.** *Let $\tau$ and $h$ be a sufficiently small numbers. Then the solutions of the difference schemes (17) satisfy the coercivity estimates*:

$$\begin{aligned} &\|\{\tau^{-2}(u_{k+1}^h - 2u_k^h + u_{k-1}^h)\}_1^{N-1}\|_{C_\tau^\alpha(C_h^\beta)} \\ &\quad \leqslant M(\alpha,\beta)[\|\varphi^{\tau,h}\|_{C_\tau^\alpha(C_h^\beta)} + \|\varphi_1^h - \varphi_{N-1}^h\|_{C_h^{\beta+m\alpha}}], \\ &\quad 0 \leqslant \alpha < 1, \ 0 < \beta + m\alpha < 1, \end{aligned}$$

*where $M(\alpha, \beta)$ does not depend on $\varphi^{\tau,h}$, $h$ and $\tau$.*

The proof of Theorem 7 is based on the abstract Theorems 3 and 4, the positivity of the operator $A_h^x$ in $C_h^\beta$ and the well-posedness of the resolvent equation of $A_h^x$ in $C_h^\beta$, $0 < \beta < 1$ and on the fact that for any $0 < \beta < \frac{1}{2m}$ the norms in the spaces $E_\beta(A_h^x, C_h)$ and $C_h^{2m\beta}$ are equivalent uniformly in $h$ (see [12,13]) and on the following theorem on the structure of the fractional spaces $E_\alpha(A^{\frac{1}{2}}, E)$.

**Theorem 8** [8]. *Let A is a strongly positive operator in a Banach space E with spectral angle* $\phi(A,E) < \frac{\pi}{2}$. *Then for* $0 < \alpha < \frac{1}{2}$ *the norms of the spaces* $E_\alpha(A^{\frac{1}{2}},E)$ *and* $E_{\frac{\alpha}{2}}(A,E)$ *are equivalent.*

## 3. Numerical analysis

We have not been able to obtain a sharp estimate for the constants figuring in the stability inequality and coercivity inequality. Therefore we will give the following results of numerical experiments of the nonlocal boundary-value problem for elliptic equation:

$$\begin{cases} -\dfrac{\partial^2 u(y,x)}{\partial y^2} - \dfrac{\partial^2 u(y,x)}{\partial x^2} = [-12y^2 + 12y - 2 + y^2(1-y)^2]\sin x, \\ 0 < y < 1, \quad 0 < x < \pi, \\ u(0,x) = u(1,x), \quad u_y(0,x) = u_y(1,x), \quad 0 \leqslant x \leqslant \pi, \\ u(y,0) = u(y,\pi) = 0, \quad 0 \leqslant y \leqslant 1. \end{cases} \tag{18}$$

The exact solution is

$$u(y,x) = y^2(1-y)^2 \sin x.$$

For approximate solutions of the nonlocal boundary-value problem (18), we will use the first and the second order of accuracy difference schemes with

Table 3.1
Numerical analysis

| $t_k \backslash x_n$ | 0 | 0.63 | 1.26 | 1.89 | 2.52 | 3.14 |
|---|---|---|---|---|---|---|
| 0.2 | 0 | 0.0150 | 0.0243 | 0.0243 | 0.0150 | 0 |
| | 0 | 0.0380 | 0.0621 | 0.0628 | 0.0392 | 0 |
| | 0 | 0.0172 | 0.0279 | 0.0281 | 0.0174 | 0 |
| 0.4 | 0 | 0.0339 | 0.0548 | 0.0548 | 0.0339 | 0 |
| | 0 | 0.0544 | 0.0905 | 0.0923 | 0.0586 | 0 |
| | 0 | 0.0343 | 0.0576 | 0.0590 | 0.0377 | 0 |
| 0.6 | 0 | 0.0339 | 0.0548 | 0.0548 | 0.0339 | 0 |
| | 0 | 0.0544 | 0.0905 | 0.0923 | 0.0586 | 0 |
| | 0 | 0.0343 | 0.0576 | 0.0590 | 0.0377 | 0 |
| 0.8 | 0 | 0.0150 | 0.0243 | 0.0243 | 0.0150 | 0 |
| | 0 | 0.0380 | 0.0621 | 0.0628 | 0.0392 | 0 |
| | 0 | 0.0172 | 0.0279 | 0.0281 | 0.0174 | 0 |
| 1.0 | 0 | 0 | 0 | 0 | 0 | 0 |
| | 0 | 0.0260 | 0.0412 | 0.0410 | 0.0248 | 0 |
| | 0 | 0.0036 | 0.0043 | 0.0036 | 0.0013 | 0 |

$\tau = \frac{1}{50}$, $h = \frac{\pi}{50}$. We have the second order difference equations with respect in $n$ with matrix coefficients. To solve this difference equations we have applied a procedure of modified Gauss elimination method. The exact and numerical solutions are given in Table 3.1.

The first line is the exact solution, the second line is the solution of the first order of accuracy difference scheme and the third line is the solution of second order of accuracy difference scheme.

Thus, the second order of accuracy difference scheme is more accurate comparing with the first order of accuracy difference scheme.

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